Investigation of LWR model with flux function driven by random free flow speed

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Abstract

The randomness is prevalent in real life. In particular, it accounts for the appearance of some especially interested phenomena in transportation system, say the spontaneous congestion. For this reason, we postulate the LWR model with stochastic setting, essentially through defining a random flux function driven by random free flow speed. Its connection to the classical LWR model is briefly discussed and properties essential for developing a flux-splitting scheme are given. A third order ENO-FD (essentially non-oscillatory finite difference) algorithm is devised and implemented to solve the postulated model numerically. We give one example based on the given model, illustrating the evaluation of predictability of traffic flow disturbance propagation in uncertain circumstance.

1 Introduction

The celebrated LWR model by [6][7] has been widely studied and plays a central role in macroscopic traffic flow simulations since its birth in 1950s. Many macroscopic models are closely connected with this model. For example, [9][10] investigate the non-equilibrium traffic flow model and [2][3] formulate the a class of first order PDE in the framework of variational theory, both with LWR model as a prototype. For general discussions on continuum traffic flow models, the reader is referred to [1][5][11][4] and the reference listed there.

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Our work is also based on LWR model but marches in another direction, and it is largely motivated by the significant randomness exhibited in the empirical data. Figure 1 presents the speed-density relation at one site from `GA400` ITS data. As is known, in traffic system the mapping from density to speed is typically multi-valued. This enables us to calculate the mean and standard deviation of speed indexed by density pointwisely. On the plot we see, the standard deviation is comparable to mean of speed in quite wide range, indicating existence of ignorable variability in velocity-density relation. This empirical feature motivates the idea of adopting the randomness by modifying the original LWR model, in the hope of replicating the reality more faithfully.

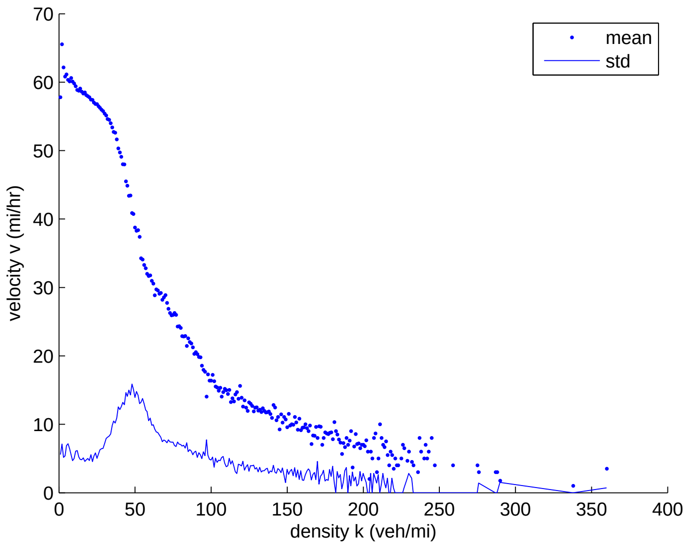

Figure 1: First and second order speed-density relation at one typical site based on GA400 ITS data

2 Formulation of LWR model in stochastic setting

Recall that the classical LWR model says that density k of traffic flow is solution of the following equation, assuming the v-k relation is time independent, i.e., system equilibrium is already achieved,

$$
\begin{aligned}
&k_t + (kv)_x = 0 \\
&v = v(k) \\
&k(x,0) = k_0(x)
\end{aligned} \tag{1}
$$

Literally, the first equation is due to the conservation law with assumption of smoothness of involved functions up to certain order, the second equation is the fundamental relation which holds under above assumption, and the last equation provides the initial state of solution. Flow $q = q(k) \equiv kv(k)$ is known as flux function of k, depicting the dynamics of concerned quantity, i.e., traffic flow density.

To correctly incorporate the randomness exhibited in equation (1), we need first to analyze the possible sources of randomness. We categorize the involved randomness in two classes. The first type of randomness is the uncertainty in data collection and processing, e.g., inaccurate reading and data roundoff, reflected in initial data setting and scatter plot of k-v relation. In this case, the collective small additive errors would altogether obey the normal distribution law, due to the central limit theorem (CLT); alternatively, if the errors are small and multiplicative, they jointly behave by the lognormal law. This type of randomness is statistical and relatively well known. The second type of randomness is due to the inherent system dynamics. Take the transportation system for example, the drivers' behavior varies from one to another, thus the group is best described in distributional terms rather than deterministically. We claim this randomness underlies random k-v relation as aforementioned. An intuitive interpretation of figure 1 would be in analogous manner as that of Brownian bridge (BB). That is, taking the v as random process indexed by k, at two points 0 and k_j the knowledge of v is relatively better, explaining the smaller variances at two ends as is shown. So based on the above analysis, we would say that the second type of randomness is essential, since the first type can usually be controlled reasonably well, through improving the measuring techniques, say.

Back to the problem of formulation LWR in stochastic setting, we would only consider the second type randomness since it is dominant and inherent. Most generally, to account for the mentioned randomness, we can express traffic speed as a positive valued multivariate function,

$$
v = v(k, \omega(x,t)) : (\mathbb{R}^+, \mathbb{R}, \mathbb{R}^+, \Omega) \mapsto \mathbb{R}^+ \tag{2}
$$

where ω is the appropriately defined set on Ω, the probability space equipped with measure $P_{x,t}(\cdot)$. This definition naturally leads to a stochastic flux function,

$$
f \equiv kv = kv(k, x, t, \omega) : (\mathbb{R}^+, \mathbb{R}, \mathbb{R}^+, \Omega) \mapsto \mathbb{R}^+ \tag{3}
$$

So substitute the formula (3) back into equation (1), we obtain the stochastic formulation of LWR,

$$K_t + (KV(K, \omega))_x = 0 \tag{4}$$

The equation (4) in general admits a random field as solution. But before attempting to obtain useful calculation results, it is desirable to justify the validity of this equation. The minimum requirement is that with trivial probability measure it is consistent with usual deterministic equation. A detailed discussion of this general form is obviously beyond the scope of current paper.

To make the analysis and computation tractable, we restrict our concern on a specific form of (4) in this paper. This is equivalent to treat a specific form of (3). In particular, we concentrate on the stochastic flux function that is independent of space and time, i.e.,

$$f = f(k, \omega) \tag{5}$$

The assumption of form (5) greatly simplifies the matter since each realization of f would be function of one variable k only. This reduces to the deterministic cases we usually face.

To address the features observed in Figure 1, we find the following form plausible. We postulate the randomness of $f(\cdot)$ is due to the uncertainty of free flow speed. First, the free flow speed is written as,

$$v_f \stackrel{d}{=} \bar{v}_f + (sk + r)\epsilon \tag{6}$$

where $\bar{v}_f$ is constant, and ϵ represents the imperfect knowledge of actual v_f, with $sk+r$ the scaling coefficient. Adopting this scaling coefficient implicitly assumes v_f depends on density k considered. With out loss of generality, we assume $E\epsilon = 0$ and $\sigma_\epsilon = 1$ (otherwise, let $\epsilon' = (\epsilon - E\epsilon)/\sigma_\epsilon$). The reason of using a function of k as scaling coefficient is as follows. If we adopt the v-k relation of,

$$(\frac{v(k)}{v_f})^\alpha + (\frac{k}{k_j})^\beta = 1 \tag{7}$$

then substituting (6) in, we have,

$$\begin{aligned} v(k) &\stackrel{d}{=} v_f(1 - (\tfrac{k}{k_j})^\beta)^{1/\alpha} \\ &= (\bar{v}_f + (sk + r)\epsilon)(1 - (\tfrac{k}{k_j})^\beta)^{1/\alpha} \end{aligned} \tag{8}$$

At this point the meaning of $sk + r$ is clear. Assuming $s > 0$, then as k increases, the variance of first term in the product of (8) becomes larger. This reflects the common perception that the free flow speed is less informative for the inference of $v(k)$ as density k increases, since in this case, the system state represented by (k, v) is getting moving away from the state $(0, v_f)$. Since v_f and k_j are symmetric in the equation (7), we may have the similar argument and term like $(sk + r)\epsilon$ accompanying k_j, such as $(\tilde{s}v + \tilde{r})\tilde{\epsilon}$. But this will result in the difficulty of expressing v as function of k explicitly. Though we can numerically solve v for given k utilizing an iterative procedure, we would rather put this issue aside at the moment and focus on relation (8) to obtain some closed-form results.

From (8) it is easy to see,

$$E(v(k)) = \bar{v}_f(1 - (\frac{k}{k_j})^\beta)^{1/\alpha} \tag{9}$$

and

$$Var(v(k)) = (sk + r)^2(1 - (\frac{k}{k_j})^\beta)^{2/\alpha} \tag{10}$$

When combining the definition of flux, we have,

$$f(k) \stackrel{d}{=} k(\bar{v}_f + (sk + r)\epsilon)(1 - (\frac{k}{k_j})^\beta)^{1/\alpha} \tag{11}$$

While the v-k relation model (8) lacks sound proof of microscopic level, it has three advantages. First, it is constructed in a heuristic manner as aforementioned. Its interpretation is quite straightforward and understandable. Second, this v-k relation is consistent with the observation that a peak of standard deviation exists between 0 and k_j, since (8) also takes a maximum value in this region. Third, this relation leads to the random flux function (11) that is easy to sample, which is actually a family of curves governed by only one random parameter ϵ.

At last, we provide the following properties that are useful in the succeeding development of numerical solution scheme,

1. For each ϵ, function $f(k)$ is smooth. It is obvious;
2. Bound of absolute value of derivative:

$$\sup_{0 \le k \le k_j} |f'(k)| \le 5k_j|\epsilon s| + 3|\epsilon r| + 3\bar{v}_f \tag{12}$$

   This is obtained by utilizing the triangle inequality. For technical convenience, we take $\epsilon \sim U(-\sqrt{3}, \sqrt{3})$, making the right hand side of (12) a finite value, denoted as α_f;
3. There exists the decomposition of $f(k)$ (namely, flux splitting):

$$f(k) = f^+(k) + f^-(k) \tag{13}$$

   where $f^+(k) = (f(k) + \alpha_f k)/2$ and $f^-(k) = (f(k) - \alpha_f k)/2$, satisfying $df^+(k)/dk > 0$ and $df^-(k)/dk < 0$.

3 The ENO-finite difference scheme with random flux function

There are two typical approaches to obtain the numerical solution of (1), i.e., the finite volume (FV) and finite difference (FD) methods. In this paper we adopt the latter since typical FV methods requires the exact value of $|f'(k)|$ or $f'(k)$, whose value is cumbersome to obtain as ϵ varies. The method we use is finite difference with ENO reconstruction. Roughly speaking, ENO (essentially non oscillatory) method reconstructs the cell boundary values through adaptively utilizing the local stencil information. In particular, the stencil with the minimal non-smoothness measure is selected. With ENO, high order finite volume (FV) or finite difference (FD) methods are immediately available. The order of accuracy depends on size of the stencil adopted. For a detailed description of this method, the reader is referred to [8] and references therein.

To sum up, we devise the algorithm as follows to repeatedly generate random flux function and solve the corresponding LWR model. This algorithm consists of two parts, i.e., initialization and time marching,

<u>* Initialization:</u>

1. Load initial data $\{k_i^0, i = 1, \ldots, N\}$, where $k_i^0 = k_0(x_i)$ is the grid point value. Moreover, set $s = 1$, $\epsilon_0 = 1$;
2. Generate ϵ_s, independent of $\epsilon_1, \ldots, \epsilon_{s-1}$ and follows $U(-\sqrt{3}, \sqrt{3})$. Generate the random flux function $f(k)$ following (11);
3. Split the flux function $f(k) = f^+(k) + f^-(k)$, following (13);

<u>* Time Marching:</u>

4. Identify $\{f^+(k_i^n), i = 1\ldots, N\}$ as cell averages and obtain $v_{i+1/2}^- = \hat{f}_{i+1/2}^+$ by ENO reconstruction. Similarly, get $v_{i+1/2}^+ = \hat{f}_{i+1/2}^-$;
5. If $n + 1 \leq T/\Delta t$, update the point value,

$$k_i^{n+1} = k_i^n - \frac{\Delta t}{\Delta x}(\hat{f}_{i+1/2} - \hat{f}_{i-1/2}) \tag{14}$$

   Let $s = s + 1$, and go back to 2; else, stop.

4 Numerical examples

Four parameters α, β, s and r are to estimate for (11), which is another topic. The parameters $\alpha = 1$, $\beta = 1$, $s = 0.05$ and $r = 3$, will be adopted afterwards in this paper, only for the purpose of illustration. Besides, we set $\bar{v}_f = 60\ mi/hr$ and $k_j = 200\ veh/mi$. To sum up, we now have the random flux function,

$$f(k) \overset{d}{=} k(60 + (0.05k + 3)\epsilon)(1 - \frac{k}{200}) \tag{15}$$

Moreover, through out the simulations in this section, we use the free boundary condition, i.e., $\partial k/\partial x|_{L,R} = 0$, where L, R indicates the left and right boundaries of computation region. We set $\Delta = 1\ sec$, Δx=0.1 mi. The speed-density relation and 20 realization of (15) are presented in Figure 2.

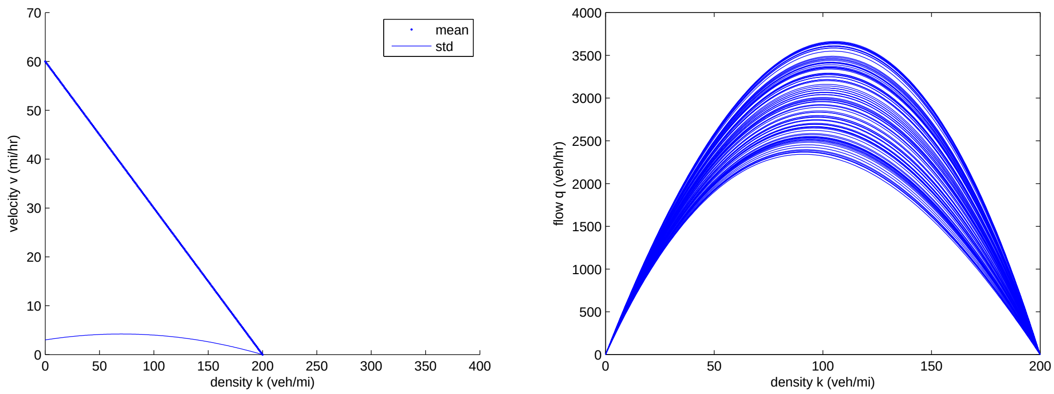

Figure 2: Hypothetical random speed-density relation (left panel) and 100 realizations of corresponding random flux function (right panel)

We first force the $\epsilon = 0$, reducing the model (15) to a deterministic one. Under the initial conditions,

$$\begin{array}{l} \text{Initial condition a: } (k_l, k_r) = (110, 30) \\ \text{Initial condition b: } (k_l, k_r) = (30, 110) \end{array} \tag{16}$$

rarefaction wave and shock wave are observed, respectively, as shown in Figure 3. We see that the behaviors predicted with the above illustrated flux function and numerical scheme are fairly as expected. In particular, the ENO type finite difference scheme does preserve the shape of shock wave well, with almost no numerical oscillations. Thus we are convinced to rely on both of the proposed random flux model and numerical scheme to investigate what is brought in with the introduce of i.i.d. random variables $\epsilon_1, \ldots, \epsilon_s$ in a series of simulations.

We then investigate the distributional properties of solution of the proposed model, which driven by the i.i.d. random sequence. In particular, the propagation of local disturbance on one lane freeway is

	Mean of k_*	Std of k_*	Mean of x_*	Std of x_*
Under initial condition a	16.09	0.50	5.99	0.75
Under initial condition b	26.46	1.92	8.63	0.63

Table 1: The statistic of numerical solution under initial condition a and b.

investigated. Here jam/vacaumm is identified by significantly larger values of local density. We assume $\epsilon_i \sim U(-\sqrt{3}, \sqrt{3})$ for the aforementioned reason. The settings for the cases studied are as follows,

1. Local jam:

$$k(x, 0) = 50 + 80 \sin((x - 2)\pi) I(2 \leq x \leq 3) \tag{17}$$

2. Local vacuum:

$$k(x, 0) = 50 - 30 \sin((x - 2)\pi) I(2 \leq x \leq 3) \tag{18}$$

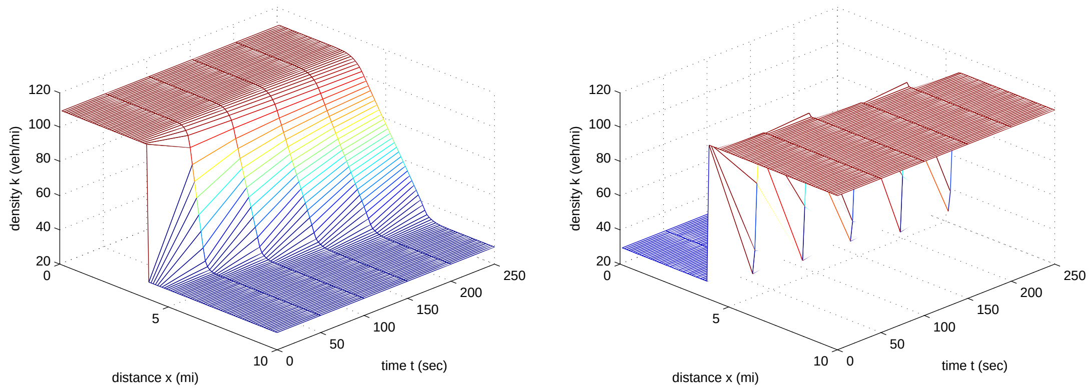

Figure 3: Temporal development of density k with $\epsilon = 0$: rarefaction waves (left panel) under Riemann initial condition $(k_l, k_r) = (110, 30)$, and shock waves (right panel) under Riemann initial condition $(k_l, k_r) = (30, 110)$.

The simulation results at $t = 600$ *sec* are shown in Figure 4. We may see with initial randomness with only variance 1, the predictability of output seems to drop significantly. Also we get statistics for the solutions, listed in Table 2. The interested quantity is selected to be the maximum local fluctuation and its location. The former is defined by,

$$k_* = \max_{0 \leq x_i \leq 10} |k(x_i) - 50| \tag{19}$$

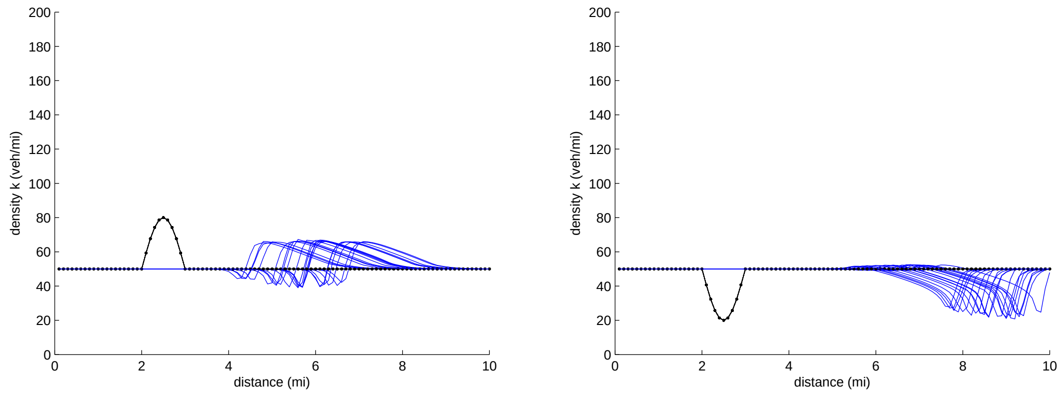

Figure 4: Propagation of local disturbance (jam: left panel; vacuum: right panel) with random flux function, $t = 600$ sec, 20 realizations. The dotted line is initial data and solid lines are simulation results.

and the latter is defined by,

$$x_* = \text{argmax}_{x_i} |k(x_i) - 50| \tag{20}$$

We calculate the coefficient of variation (CoV) using the values listed in Table 2 and take its reciprocal as the measure of predicting accuracy. We find, $CoV_{a,k_*} > CoV_{b,k_*}$ and $CoV_{a,x_*} > CoV_{b,x_*}$. This result can be interpreted that travel of jam is more unpredictable. We remark that such result can not be obtained from a deterministic simulation.

5 Conclusion and remarks

In this paper, the random flux function driven by the stochastic free flow speed is postulated to address the variability features of data. We discuss the sources of randomness and the random flux function in a general context, present and derive a specific form of stochastic flux function, we also provide essential properties of this function that for developing a flux splitting scheme. A ENO-FD type numerical scheme is devised and implemented for solution of this proposed model. At last, we present one example to illustrate how the predictability of two types of local density disturbance travel can be evaluated, based on the postulated random LWR model. We see, with the proposed random flux function, probabilistic investigation of the transportation issues are enabled.

We remark that the model parameters need to estimate to achieve realistic simulation result, which is a gap left by the current work.

Moreover, we assume the random series ϵ's that drive the flux function are independent. Taking into consideration of temporal correlation of the series may provide new insight.

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